

### 3.4. Separation of Variables — Spherical system

General Equation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{K} g(r, \theta, \phi, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

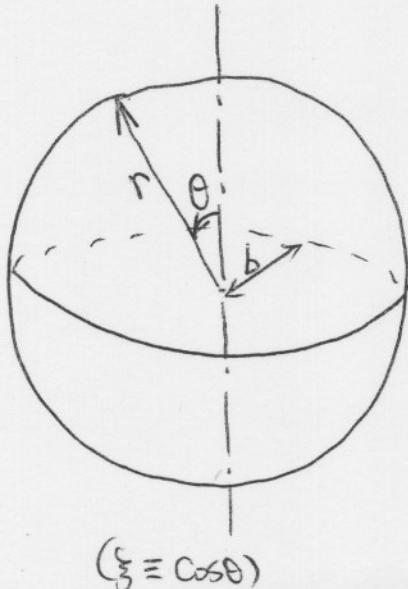
With new independent variable:  $\xi = \cos \theta$

$$\underbrace{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[ (1-\xi^2) \frac{\partial T}{\partial \xi} \right] + \frac{1}{r^2 (1-\xi^2)} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{K} g(r, \xi, \phi, t)}_{= \frac{1}{\alpha} \frac{\partial T}{\partial t}}$$

\* Example 1.

Find an expression for the temperature distribution  $T(r, \xi, t)$  in a solid sphere  $0 \leq r \leq b$  which is initially at temperature  $F(r, \xi)$  and for times  $t > 0$  boundary surface is kept at  $T = 0$ . (no heat generation)

The complete problem: 2D,  $T = T(r, \xi, t)$



$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[ (1-\xi^2) \frac{\partial T}{\partial \xi} \right] = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

B.C.  $\left. T \right|_{r=0} = \text{finite}$   
 $\left. T \right|_{r=b} = 0$

$\left. T \right|_{\xi=\pm 1} = \text{finite}$  ( $\theta = 0$  and  $\pi$ ) Natural B

T.C.  $\left. T \right|_{t=0} = F(r, \xi)$

To solve the equation, we can define a new dependent variable:  $V(r, \xi, t) \equiv \underbrace{r^{\frac{1}{2}} T(r, \xi, t)}$

$$\text{so: } \begin{cases} \frac{\partial T}{\partial r} = r^{-\frac{1}{2}} \frac{\partial V}{\partial r} - \frac{1}{2} r^{-\frac{3}{2}} V & (T = r^{\frac{1}{2}} V) \\ \frac{\partial^2 T}{\partial r^2} = r^{-\frac{1}{2}} \frac{\partial^2 V}{\partial r^2} - r^{-\frac{3}{2}} \frac{\partial V}{\partial r} + \frac{3}{4} r^{-\frac{5}{2}} V \end{cases}$$

Therefore, the equation for  $V(r, \xi, t)$  becomes:

$$\boxed{\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{1}{4} \frac{V}{r^2} + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[ (r\xi)^2 \frac{\partial V}{\partial \xi} \right] = \frac{1}{2} \frac{\partial V}{\partial t}}$$

B.C.  $\left. \begin{array}{l} V|_{r=0} = 0 \\ V|_{r=b} = 0 \end{array} \right\}$  homogeneous!

$\left. V \right|_{\xi=\pm 1} = \text{finite} \leftarrow \text{Natural B.C.}$

I.C.  $\left. V \right|_{t=0} = r^{\frac{1}{2}} F(r, \xi)$

① Separation of  $V(r, \xi, t)$ .

Let:  $V(r, \xi, t) = R(r) H(\xi) T(t)$

Then:  $R'' H T + \frac{1}{r} R' H T - \frac{1}{4r^2} R H T + \frac{1}{r^2} \frac{d}{d\xi} \left[ (r\xi)^2 \frac{dH}{d\xi} \right] R T = \frac{1}{2} R H \frac{dT}{dt}$

Each side divided by  $R \Theta \Gamma$ :

$$\underbrace{\frac{R'' + \frac{1}{r}R' - \frac{1}{4r^2}R}{R}}_{\text{function of } r \text{ and } \xi} + \underbrace{\frac{\frac{1}{r^2} \frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta}}_{\text{(H)}} = \frac{\frac{1}{\alpha} \frac{d\Gamma}{dt}}{\Gamma} = -\lambda^2$$

$\downarrow$

$$\Gamma(t) = E e^{-\lambda^2 \alpha t}$$

For "R" and " $\Theta$ ":

$$\underbrace{\frac{R'' + \frac{1}{r}R' - \frac{1}{4r^2}R}{R}}_{\text{R}} + \lambda^2 = -\frac{1}{r^2} \underbrace{\frac{\frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta}}_{\text{(H)}}$$

Multiplying each side by  $r^2$ :

$$\underbrace{\frac{r^2 R'' + r R' - \frac{1}{4} R + \lambda^2 r^2 R}{R}}_{\text{function of } r \text{ only}} = -\underbrace{\frac{\frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta}}_{\text{function of } \xi \text{ only}} = \mu$$

Then, the equation for R and  $\Theta$  becomes:

$$\begin{cases} \frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right] + \mu \Theta = 0 \\ r^2 R'' + r R' + \left[ \lambda^2 r^2 - \frac{1}{4} - \mu \right] R = 0 \end{cases}$$

(2) Solving ODEs

(1) For  $\Theta(\xi)$ :

$$\frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right] + \mu \Theta = 0$$

B.C.  $\Theta|_{\xi=\pm 1} = \text{finite}$

The natural B.C. at  $\xi = \pm 1$  ( $\theta = 0$  and  $180^\circ$ ) requires  $\mu$  can only take certain values (eigenvalues).

$$\underbrace{\mu = n(n+1)}_{n=0, 1, 2, \dots}$$

so:  $\frac{d}{d\xi} \left[ (1-\xi^2) \frac{d\Theta}{d\xi} \right] + n(n+1) \Theta = 0$

i.e.:  $(1-\xi^2) \Theta'' - 2\xi \Theta' + n(n+1) \Theta = 0$   $\leftarrow$  Legendre equation

General solution:  $\Theta_n(\xi) = C P_n(\xi) + D Q_n(\xi)$   $P_n, Q_n$ : Legendre func

B.C.: Because  $Q_n(\xi)$  diverges at  $\xi = \pm 1 \Rightarrow D = 0$

i.e.:  $\boxed{\Theta_n(\xi) = C P_n(\xi)}$   $n=0, 1, 2, \dots$

(2) For  $R(r)$ :

$$r^2 R'' + r R' + \left[ \lambda^2 r^2 - \frac{1}{4} - n(n+1) \right] R = 0$$

i.e.:  $\boxed{R'' + \frac{1}{r} R' + \left[ \lambda^2 - (n+\frac{1}{2})^2 \frac{1}{r^2} \right] R = 0}$   $\leftarrow$  Bessel Equation of order  $(n+\frac{1}{2})$ !

general solution:  $R(r) = A J_{n+\frac{1}{2}}(\lambda r) + B Y_{n+\frac{1}{2}}(\lambda r)$

Imposing B.C.  $R|_{r=0} = \text{finite}$ :

$B=0$  because  $Y_{n+\frac{1}{2}}(\lambda r) \rightarrow \infty$  at  $r=0$ .

So:  $R(r) = A J_{n+\frac{1}{2}}(\lambda r)$

Imposing B.C.  $R|_{r=b} = 0$ :

then:  $R|_{r=b} = A J_{n+\frac{1}{2}}(\lambda b) = 0$

i.e.:  $\underbrace{J_{n+\frac{1}{2}}(\lambda b)}_{} = 0$  (for each  $n$ )

so  $\lambda$  can only take certain values:  $\lambda = \lambda_{nm}$ ,  $m=1, 2, \dots$   
(eigenvalues)

and for each  $n$  and  $m$ :

$$\boxed{R(r) = A J_{n+\frac{1}{2}}(\lambda_{nm} r)} \quad \left\{ \begin{array}{l} n=0, 1, 2, \dots \\ m=1, 2, 3, \dots \end{array} \right.$$

③ Making final solution:

For each  $n$  and  $m$ :

$$\boxed{V_{nm}(r, \xi, t) = C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) e^{-\lambda_{nm}^2 \alpha t}}$$

and:

$$\underbrace{V(r, \xi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) e^{-\lambda_{nm}^2 \alpha t}}$$

(4) Determining unknown coefficient.

Applying initial condition.  $V|_{t=0} = r^{\frac{1}{2}} F(r, \xi)$

$$\text{then: } r^{\frac{1}{2}} F(r, \xi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi)$$

operate on both side by operators:  $\int_{\xi=-1}^{+1} P_n'(\xi) d\xi$   
 $\int_{r=0}^b J_{n+\frac{1}{2}}'(\lambda_{nm} r) r dr$

and use orthogonal properties for  $P_n(\xi)$  and  $J_{n+\frac{1}{2}}(\lambda_{nm} r)$ :

$$C_{nm} = \frac{\int_{r=0}^b \int_{\xi=-1}^{+1} r^{\frac{3}{2}} J_{n+\frac{1}{2}}(\lambda_{nm} r) P_n(\xi) F(r, \xi) d\xi dr}{\underbrace{\int_{\xi=-1}^{+1} [P_n(\xi)]^2 d\xi}_{\frac{2}{2n+1}} \underbrace{\int_{r=0}^b J_{n+\frac{1}{2}}^2(\lambda_{nm} r) r dr}_{\frac{b^2}{2} [J_{n+\frac{1}{2}}'(\lambda_{nm} b)]^2}}$$

Note:  $T(r, \xi, t) = r^{-\frac{1}{2}} V(r, \xi, t)$

The transformation from  $T$  to  $V$  is necessary for two cases:  
 $T = T(r, \xi, t)$  or  $T(r, \xi, \phi, t)$ .

\* Example 2. — 1D problem [ $T = T(r, t)$ ]

The transient heat conduction problem for a sphere involving  $(r, t)$  variables only can be transformed into a problem of a slab or a semi-infinite medium.

Case 1: solid sphere ( $0 \leq r \leq b$ ).

The complete problem (general):

$$\left| \begin{array}{l} \frac{1}{r} \frac{\partial^2 (rT)}{\partial r^2} + \frac{1}{k} g(r) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \\ \text{B.C. } \left. T \right|_{r=0} = \text{finite} \\ \left. \frac{\partial T}{\partial r} + HT \right|_{r=b} = f \\ \text{I.C. } \left. T \right|_{t=0} = F(r) \end{array} \right.$$

$$(0 < r < b)$$



Define a new dependent variable:

$$\underline{U}(r, t) \equiv r T(r, t) \quad (T = \frac{1}{r} U)$$

Therefore:

$$\left| \begin{array}{l} \frac{\partial^2 U}{\partial r^2} + \frac{1}{k} g(r) = \frac{1}{\alpha} \frac{\partial U}{\partial t} \\ \text{B.C. } \left. U \right|_{r=0} = 0 \\ \left. \frac{\partial U}{\partial r} + (H - \frac{1}{b}) U \right|_{r=b} = b f \\ \text{I.C. } \left. U \right|_{t=0} = r F(r) \end{array} \right. \quad (0 < r < b)$$

This is a problem of conduction in a slab.

Case 2: Hollow sphere ( $a \leq r \leq b$ ).

The complete problem (general):

$$\boxed{\begin{aligned} & \frac{1}{r} \frac{\partial^2 (rT)}{\partial r^2} + \frac{1}{k} g(r) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \\ \text{B.C. } & \left. \left( -\frac{\partial T}{\partial r} + H_1 T \right) \right|_{r=a} = f_1 \\ & \left. \left( \frac{\partial T}{\partial r} + H_2 T \right) \right|_{r=b} = f_2 \\ \text{I.C. } & T|_{t=0} = F(r) \end{aligned}}$$

$$(a < r < b)$$



Define a new dependent variable:

$$\underline{U(r,t) \equiv rT(r,t)} \quad (\bar{T} = \frac{1}{r} U)$$

Therefore

$$\boxed{\begin{aligned} & \frac{\partial^2 U}{\partial r^2} + \frac{1}{k} g(r) = \frac{1}{\alpha} \frac{\partial U}{\partial t} \quad \leftarrow a < r < b \\ \text{B.C. } & \left. \left( -\frac{\partial U}{\partial r} + \left( H_1 + \frac{1}{a} \right) U \right) \right|_{r=a} = af_1 \\ & \left. \left( \frac{\partial U}{\partial r} + \left( H_2 - \frac{1}{b} \right) U \right) \right|_{r=b} = bf_2 \\ \text{I.C. } & U|_{t=0} = rF(r) \end{aligned}}$$

A shift in space coordinate is necessary: define  $\underline{x \equiv r-a}$ .



So, the equation for  $U(x, t)$  becomes.

$$\boxed{\frac{\partial^2 U}{\partial x^2} + \frac{x+a}{k} g(x+a) = \frac{1}{\alpha} \frac{\partial U}{\partial t} \quad \leftarrow 0 < x < (b-a)}$$

B.C.  $\left. \begin{array}{l} -\frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial x} \end{array} \right|_{x=0} + \left. \begin{array}{l} (H_1 + \frac{1}{a})U \\ (H_2 - \frac{1}{b})U \end{array} \right|_{x=(b-a)} = \left. \begin{array}{l} af_1 \\ bf_2 \end{array} \right.$

I.C.  $U \Big|_{t=0} = (x+a) F(x+a)$

This is a problem of conduction for a slab.